Velocity statistics in dissipative, dense granular media

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We use a two-dimensional random-force model to investigate the velocity distributions in driven granular media. In general, the shape of the distribution is found to depend on the degree of dissipation and the packing fraction but, in highly dissipative systems, the velocity distributions have tails close to exponential. We show that these arise from the dynamics of single particles traveling in dilute regions and influenced predominantly by the random force. A self-consistent kinetic theory is developed to describe this behavior.

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Driven granular materials are simple examples of non-equilibrium statistical systems. Recently, there has been much interest in characterizing the velocity statistics of such systems [1,2]. It has been observed both experimentally and in simulation that the velocity distributions deviate strongly from the Maxwell-Boltzmann statistics applicable in thermal equilibrium. However, to date, no consistent picture has emerged.

Much of the previous work has concentrated on excited systems in which the granular bed is fluidized. Under such conditions, non-Gaussian behavior has been observed in experiments which include granular flows [3], vibrated beds [4], and sheared [5] and electrostatically driven [6] systems. Non-Gaussian velocity distributions have also been observed in simulation [7,8] and several kinetic theories have been developed to explain this behavior [9–11]. One approach uses uniform heating of the grains and collisional dissipation to obtain a distribution for the velocities v of the form $P(v) \sim \exp(-\alpha|v|^{1.5})$ [9]. Alternatively, a Maxwell model for which the collision rate is particle velocity independent offers an approximate solution of the form $\exp(-\alpha|v|)$ in the tails of the probability distribution [11].

One of the main assumptions of these kinetic theories is that the system is spatially homogeneous. In many real systems, the inelastic nature of particle collisions induces inhomogeneity which may lead to clustering. That there is a connection between bed density and velocity statistics is suggested by the observation that the distributions appear to vary with the height at which they are measured in a vertically vibrated granular bed [12]. However, how dissipation, bed density and spatial inhomogeneity influence the velocity distribution is still unclear.

In this Brief Report we discuss the velocity statistics in dissipative, dense granular media. We consider a simple two-dimensional random-force model for a collection of inelastic particles. In the limit of high dissipation, this model exhibits exponential velocity tails for a wide range of the parameters. Within this model, we show that it is the motion of particles moving in less dense regions and influenced predominantly by the random force which leads to the exponential tails. We develop a self-consistent single particle kinetic theory to describe the behavior of the high energy particles.

Our simulations use two-dimensional soft-sphere molecular dynamics to investigate the particle motion. The interactions between particles are modeled using a linear spring-dashpot force in the normal direction and a tangential sliding frictional force. For simplicity, particle rotation is ignored. The spherical grains were chosen to have a diameter of 3 mm and a density of 2500 K g m^{-3}. The spring constant was chosen to be 5000 Nm^{-1} while the friction coefficient μ was chosen to be either 0 or 0.5 and the coefficient of restitution ε was varied within the range 0.05 to 0.95.

To excite the system, each particle is influenced by a random force [13]. The model may be thought of as a method of describing a single layer in a dense three-dimensional granular bed, the random force representing the interactions between different layers induced by vibration [14]. The N particles are assumed to move on a square horizontal plane with sides of extent $L=0.2$ m. The particles are confined by reflective walls, the wall-particle restitution coefficient being the same as the particle-particle coefficient. Collisions dissipate energy, while energy is added through the random excitation of each particle. The equation of motion along the plane for particle i is therefore

$$m\frac{d\mathbf{v}_i}{dt} = \sum_{j\neq i} \mathbf{F}_{i,j} + \eta_i(t), \tag{1}$$

where m is the particle mass, $\mathbf{F}_{i,j}$ is the interaction force between particles i and j, $v_{i\alpha}$ the particles' velocity component in Cartesian direction α, and $\eta_{i\alpha}(t)$ is Gaussian white noise with correlator $\langle \eta_{i\alpha}(t)\eta_{j\beta}(t')\rangle = 2D\delta(t-t')\delta_{i,j}\delta_{\alpha,\beta}$. We set the overall velocity scale to a realistic magnitude by choosing $D=1\times 10^{-8}$ N^2 s.

We have investigated the velocity statistics for a wide range of parameter values. To remove the fluctuations of the centre of mass of the system, the velocity of the centre of mass is subtracted from each particle's velocity when calculating the statistics. In general, the distributions may deviate strongly from Gaussian. However, there is not a simple form that fits all the distributions but rather a set of curves that become closer to pure exponentials in the limit of high dissipation and packing fraction [8]. In Fig. 1, the solid lines show the velocity statistics corresponding to two extreme cases. Figure 1(a) is for a low dissipation system, having a smaller number of particles, a coefficient of restitution of 0.95 and no tangential friction. The velocity statistics are close to Gaussian in this case. Figure 1(b) shows the corresponding data for a highly dissipative system. Here we have simulated a large number of particles, each with a coefficient

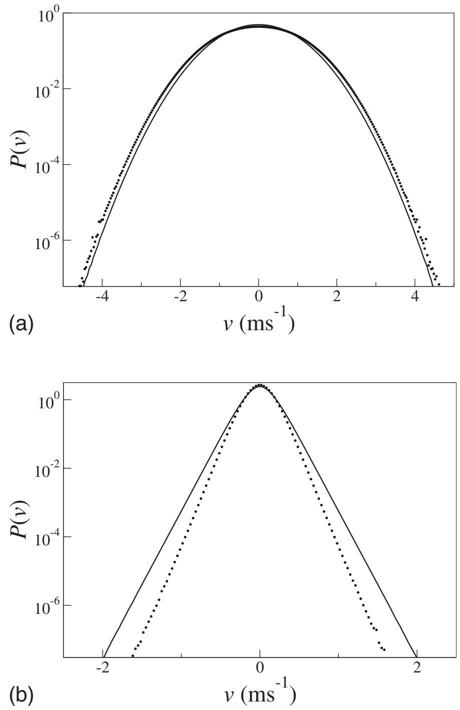

FIG. 1. The velocity distribution for the 2D simulation of a system of N particles with diameter $d=3$ mm, contained in a square region with $L=0.2$ m. The solid lines show the full velocity distribution while the dots show the post-collisional velocity statistics, as described in the text. The parameter values we have used to generate the two figures are (a) $N=1500$, $\varepsilon=0.95$, $\mu=0.0$ and (b) $N=3500$, $\varepsilon=0.1$, $\mu=0.5$.

of restitution of 0.1 and a friction coefficient of 0.5. In this case the distribution is far from Gaussian and exhibits exponential high velocity tails. In general we find that, as the number of particles increases, the velocity distributions become closer to exponential. Similarly, reducing ε or increasing the friction coefficient μ also forces the velocity distribution to become exponential in the tails.

We now ask, within these highly dissipative systems, how do the particles with the highest speeds get their energy? There are two possible mechanisms. The particles could gain their energy either from the random force or from particle-particle collisions. To determine which, if either, of these two possibilities dominates, we consider the post-collisional velocity statistics, shown by the dotted curves in Fig. 1. These curves are generated by only binning the velocities of particles that have just had a collision in the previous timestep. Figure 1(a) shows that for the weakly dissipative system there is very little difference between the post-collisional velocity distribution and the full velocity distribution. However, in the highly dissipative case, Fig. 1(b), there is a clear distinction between the distributions in the tails; the post-collisional velocity distribution falls steadily away from the full distribution as the velocity increases. In the limit of extremely high velocities, the post-collisional velocity of these particles becomes increasingly irrelevant. This implies that the the high velocity tails are generated by particles that are driven predominantly by the random force. Can we therefore understand the high velocity tails of the highly dissipative systems by focusing on individual particles?

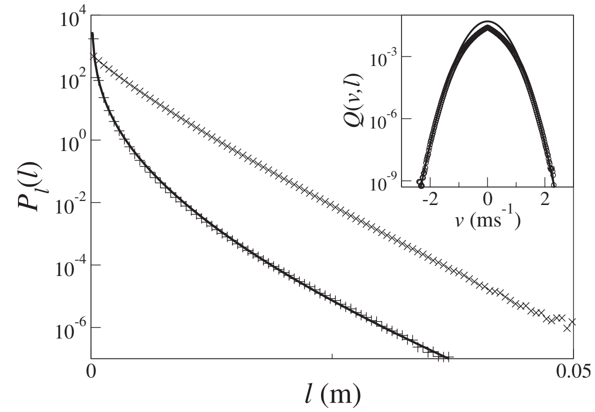

FIG. 2. The distribution of free paths between collisions l in the 2D many-body simulation for particles of diameter 3 mm, in a system with $L=0.2$ m and $N=1500$, $\varepsilon=0.95$, $\mu=0.0$ (upper) and $N=3500$, $\varepsilon=0.1$, $\mu=0.5$ (lower). The line fit is a function of the form $Al^{-\gamma}\exp(-Bl^{\beta})$ with $\beta=2/3$. The inset shows the velocity statistics for a single, randomly forced particle with noise strength $D=1\times10^{-8}$ N^2 s contained within a circle of radius $l=0.01$ m. The solid line shows a Gaussian fit to the data.

To do this, we have investigated a simple single-particle model which is able to reproduce the high energy tails of the velocity statistics. We represent a particle moving between collisions by a particle traveling within a circle of radius l. This distance represents the free path length. The particle experiences a random force and moves until it reaches the boundary circle. The particle is then placed *at rest* in the centre of a circle with a new radius l', chosen from the probability distribution of free paths obtained from the 2D many-body simulations. The process is repeated very many times and the velocity distribution is determined by sampling the instantaneous velocity of the particle uniformly in time [15].

This simple model is expected to capture the physics of the high velocity particles driven solely by the random force. The main approximation used is that the particle starts from rest each time. This approximation becomes more valid in strongly dissipative systems where we have shown the post-collisional velocity to be increasingly irrelevant in the tails of the distribution. A second approximation in the single particle model is that the particle is stopped the *first* time it reaches a distance l. Again, this approximation becomes more valid as the particle's velocity increases.

Two distributions of free paths between collisions that we have used in the single particle model are shown in Fig. 2; they have been obtained from the many-body simulations by measuring the distance a particle has moved between two consecutive collisions with other particles. The system parameters are the same as have been used to obtain Figs. 1(a) and 1(b). Figure 3 compares the results for the single particle model with the velocity distributions obtained using the full 2D many-body simulations. The comparison made is for the same sets of parameter values as for the curves shown in Figs. 1 and 2. For the highly dissipative system, the slopes of the high velocity tails given by the two models are almost identical. However, the model gives different predictions at low velocities, as is expected. Consequently, normalized

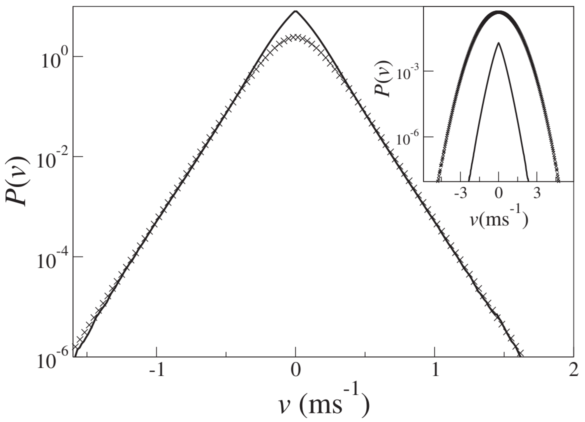

FIG. 3. Comparison of the single particle model (line) against the 2D many-body simulation for the highly dissipative system (crosses). Here we have scaled the single particle results *vertically* so that they overlay that of the simulation, as described in the text. The parameters are those used in Fig. 2. The inset shows the corresponding comparison for the weakly dissipative case. The curves are of quite different form and no attempt has been made to scale.

curves would not fall on top of each other in the high velocity limit. To emphasize the very similar slopes of the distributions at high velocities we have displaced the curves vertically, corresponding to slightly different normalizations. It is seen that the agreement between the two models is remarkably good over many decades of $P(v)$. For completeness, we show as an inset in Fig. 3 the corresponding comparison for the low dissipation system. In this case the single particle model clearly fails, as is to be expected.

The numerical procedure of the single particle model can be described analytically by the following integral equation:

$$P(v) = \int_0^\infty Q(v,l)P_l(l)dl. \tag{2}$$

Here $P(v)$ is the velocity distribution, $P_l(l)$ is the distribution of free paths between collisions, l, obtained from the 2D simulation (Fig. 2) and $Q(v,l)$ is the velocity distribution for a randomly accelerated particle within a circle of fixed radius l. Equation (2) states that the probability distribution of velocities for a collection of particles can be obtained from the velocity distributions for a single particle contained in a circle of radius l, weighted by the probability of a free path of length l occurring. The velocity distribution can be accurately constructed in this way over many decades of $P(v)$, as we have already shown numerically in Fig. 3.

To enable an analytic treatment of Eq. (2), the probability distribution $Q(v,l)$ can readily be obtained from simulation of a single particle within a fixed-size circle. The data can be fitted asymptotically to a Gaussian of the form

$$Q(v,l) \sim \exp(-cv^2/l^{2/3}), \tag{3}$$

as is shown in the inset of Fig. 2. Here c is a numerical constant which depends upon D and m. If we make the general assumption that the tails of the velocity and free path probability distributions have the forms $P(v) \sim \exp(-a|v|^\alpha)$ and $P_l(l) \sim \exp(-bl^\beta)$, we may use Eqs. (2) and (3) to obtain the relationship between α and β,

$$\alpha = \frac{6\beta}{3\beta+2}. \tag{4}$$

This relationship does not depend upon the constants a, b and c or on any nonexponential prefactors to $P(v)$ and $P_l(l)$.

In order to determine α and β explicitly, we apply a simple generalization of kinetic theory to the case of a randomly accelerated particle. We have shown that the high velocity tails arise from single particle motion driven by the random force. In order for this to occur, a particle must have free space to move. Consequently, the high velocity particles are found in less dense regions of the system, as we have confirmed by visual inspection. We suppose that within a dilute region, the system behaves as a homogeneous dilute "gas" of randomly accelerated particles, and that correlations between these particles can be ignored.

Let $S(l)$ be the probability that the particle has survived a distance l without colliding with any other particle. The probability of colliding within some small distance dl equals the time taken to cross the region multiplied by the mean collision rate. Consequently, $S(l)$ obeys the equation

$$S(l+dl) = S(l)\left[1 - \frac{f}{v}dl\right], \tag{5}$$

where v is the velocity of the particle and f is the mean collision frequency. If v is constant, Eq. (5) can be solved to give simple exponential decay as in standard kinetic theory [16]. However, for a randomly accelerated particle, v is no longer constant. The typical particle velocity grows with time as $t^{1/2}$ while the typical distance traveled over this time grows as $t^{3/2}$. Substituting $v \sim l^{1/3}$ into Eq. (5) gives the stretched exponential $S(l) \sim \exp(-bl^{2/3})$, where b is a constant. The corresponding distribution of free paths between collisions $P_l(l)$ is related to $S(l)$ by $P_l(l) = -dS/dl$, and is therefore also a stretched exponential with exponent $\beta = 2/3$. This result, together with Eq. (4), implies that $\alpha = 1$. Both these values are consistent with the numerical simulation results for the higher dissipation systems described above. Figures 1(b) and 3 show, for the high dissipation system, exponential behavior of the velocity distribution corresponding to $\alpha = 1$, while in Fig. 2 the related data for the free path distribution have been fitted to a stretched-exponential with $\beta = 2/3$, the continuous line.

The agreement between the many-body simulations and the single particle model suggests the following mechanism for generating non-Gaussian velocity distributions in dense dissipative granular systems. The high velocity particles achieve their velocity predominantly through the action of the noise within less dense regions. The noise induced acceleration gives rise to stretched exponential distributions for the paths between collisions. This in turn leads to non-Gaussian behavior of the velocity distribution.

To conclude, we have demonstrated that a two-dimensional random force model exhibits exponential behavior of the velocity distribution in the high dissipation limit. We have identified the key physical mechanisms responsible for the high velocity tails and have developed a self-consistent kinetic theory able to describe this behavior. The success of this theory results from the separation of velocity scales between the highly active particles and the slow moving dense background of particles. It will be interesting to investigate whether a similar mechanism is at work in the wide range of driven granular systems that exhibit anomalous velocity statistics.

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